[06-03-31-T11]

Compound interest

■ Simplest case

Deposit a dollars. Receive interest at rate r, $r > 0$, per quarter.

Note: $a + ra = a(1 + r)$

■ Value of account at end of each quarter.

Quarter	Value at end quarter	Value simplified
$k = 1$	$a(1+r)$	$a(1+r)$
$k = 2$	$a(1+r)(1+r)$	$a(1+r)(1+r)^1$
$k = 3$	$a(1+r)(1+r)(1+r)$	$a(1+r)(1+r)^2$
$k = 4$	$a(1+r)(1+r)(1+r)(1+r)$	$a(1+r)(1+r)^3$
$k = 5$	$a(1+r)(1+r)(1+r)(1+r)(1+r)$	$a(1+r)(1+r)^4$
$\vdots$	$\vdots$	$\vdots$
$k = n-2$		$a(1+r)(1+r)^{n-3}$
$k = n-1$		$a(1+r)(1+r)^{n-2}$
$k = n$		$a(1+r)(1+r)^{n-1}$

Therefore, $V_n = a(1+r)(1+r)^{n-1} = a(1+r)^n$ is the value of the account at end of nth quarter.

[EX1] Deposit \$100 at 5% interest compounded quarterly.

At end of 1st quarter, the value of account is $100(1.05) = \$105$.

At end of 2nd quarter, the value of account is $100(1.05)^2 = \$110.25$.

At end of 3rd quarter, the value of account is $100(1.05)^3 = \$115.77$.

At end of 4th quarter, the value of account is $100(1.05)^4 = \$121.56$.

■ Generality of $A_n = a(1 + r)^n$

This result applies to many phenomena in finance, physics, chemistry, and other disciplines. Any case in which the k^{th} value of a function becomes the input for computing the $(k+1)^{\text{th}}$ value. The amount of a radioactive material as a function of time is $A_t = A_0(1+r)^t$, A_0 is the original amount, $-1 < r < 0$. Washing your hands is an interesting case. Suppose with each washing you eliminate $\frac{1}{2}$ of the contaminate; then, the amount of contaminate remaining after n successive washings is $A_n = A_0(1+r)^n$, $-1 < r < 0$. I hope this does not lead to obsessive behavior.

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■ Slightly more complicated case

Deposit a dollars at beginning of every quarter. Receive interest at rate r, $r > 0$, per quarter.

■ Value of account at end of each quarter.

Quarter	Value at end quarter	Value simplified
$k = 1$	$a(1+r)$	$a(1+r)$
$k = 2$	$[a + a(1+r)](1+r)$	$a(1+r) + a(1+r)^2$
$k = 3$	$[a + a(1+r) + a(1+r)^2](1+r)$	$a(1+r) + a(1+r)^2 + a(1+r)^3$
$\vdots$	$\vdots$	$\vdots$
$k = n$		$a(1+r) + a(1+r)^2 + a(1+r)^3 + \ldots + a(1+r)^n$

Therefore, $V_n = a(1+r) + a(1+r)^2 + a(1+r)^3 + \ldots + a(1+r)^n$ is the value at end of nth quarter. Since this is a geometric series whose 1st term is $a(1+r)$ and whose common ratio is $(1+r)$, we may rewrite it:

$$V_n = \frac{a(1+r)[1-(1+r)^n]}{1-(1+r)} = \frac{a(1+r)[(1+r)^n-1]}{r}.$$

[EX1] Deposit \$100 at beginning of each quarter. 5% interest compounded quarterly. Round values to nearest \$0.01.

At end of 1st quarter, the value of account is $\frac{100(1.05)(1.05^1-1)}{.05} = \$105.$.

At end of 2nd quarter, the value of account is $\frac{100(1.05)(1.05^2-1)}{.05} = \$215.25$.

At end of 3rd quarter, the value of account is $\frac{100(1.05)(1.05^3-1)}{.05} = \$331.01$.

At end of 4th quarter, the value of account is $\frac{100(1.05)(1.05^4-1)}{.05} = \$452.56$.

■ The derivation in the text on page 75 is nicer.